Reference frames which separately store non-commuting conserved quantities

Sandu Popescu,[1,*] Ana Belén Sainz,[2,3,†] Anthony J. Short,[1,‡] and Andreas Winter[4,5,§]

[1] *H. H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol, BS8 1TL, United Kingdom*
[2] *International Centre for Theory of Quantum Technologies, University of Gdańsk, 80-308 Gdańsk, Poland*
[3] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada, N2L 2Y5*
[4] *ICREA—Institució Catalana de Recerca i Estudis Avançats, Pg. Lluis Companys 23, 08010 Barcelona, Spain*
[5] *Física Teòrica: Informació i Fenòmens Quàntics, Departament de Física, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain*

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Even in the presence of conservation laws, one can perform arbitrary transformations on a system if given access to a suitable reference frame, since conserved quantities may be exchanged between the system and the frame. Here we explore whether these quantities can be separated into different parts of the reference frame, with each part acting as a 'battery' for a distinct quantity. For systems composed of spin-$\frac{1}{2}$ particles, we show that the components of angular momentum S_x, S_y and S_z (non-commuting conserved quantities) may be separated in this way, and also provide several extensions of this result. These results also play a key role in the quantum thermodynamics of non-commuting conserved quantities.

Conservation laws are amongst the most important and widely used aspects of physics, greatly restricting the possible transformations that an isolated system can undergo [1–3]. However, when a system is not isolated but allowed to interact with other systems, with conservation laws applying only globally, much greater freedom is possible [4–9]. The situation is particularly interesting in quantum theory, where different conserved quantities may not commute (such as the different components of angular momentum), and interference effects are crucial.

Perhaps surprisingly, it has been shown that any transformation of a quantum system can be implemented, as long as one has access to an appropriate ancillary system [6–9]. This additional system plays a dual role of providing a reference frame for the transformation, and acting as a reservoir which can exchange conserved quantities with the system.

While undergoing its transformation, the system will generally exchange many different conserved quantities with the reference frame. An interesting question is whether the exchange of each of these different conserved quantities can be separated into different parts of the reference system, with each part effectively acting as a 'battery' for a specific conserved quantity. When the conserved quantities commute, this is possible [10]. The main question we raise - and answer - in the present paper, is whether this is possible when the conserved quantities do *not* commute. In this situation one might expect that it is impossible to separate the various conserved quantities into different reservoirs, because one cannot even measure one of them without disturbing the other. Surprisingly, we show that this is possible. Moreover, after the quantities have been separated in this way, (commuting) measurements could be made on each component of the reference system, the expectation values of which would give the average changes in each conserved quantity, despite these quantities not commuting on the original system.

As a concrete example, consider rotations of a spin-$\frac{1}{2}$ particle in the presence of angular momentum conservation. We would like to be able to localise any changes in the three components of spin s_x, s_y and s_z of the spin-$\frac{1}{2}$ particle within different subsystems in the reference frame (see Fig. 1). In this paper, we show that this is indeed possible. Our approach will generalise to any unitary transformation on any number of spin-$\frac{1}{2}$ systems. Furthermore, as an interesting example of such a protocol, we show how to completely extract the three components of angular momentum of an unknown spin state into three distinct systems (up to arbitrary precision). Finally, in the Supplementary material [11, Sec. II], we extend these results for arbitrary conserved quantities of any system of dimension 2^n and to conservation of angular momentum under rotations for arbitrary spin.

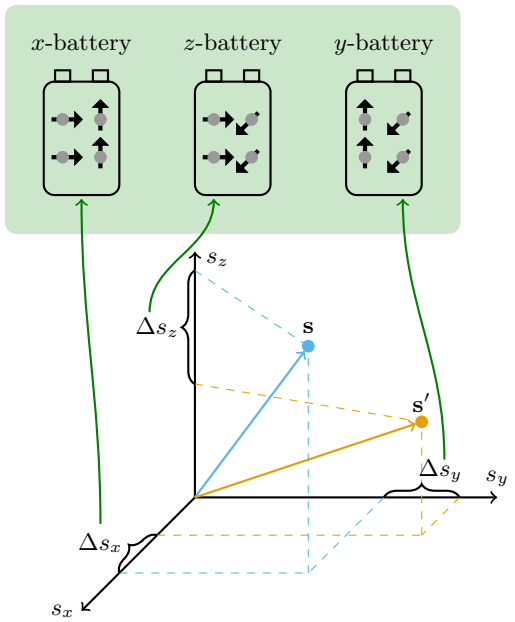

FIG. 1. A spin-$\frac{1}{2}$ system, initially with spin $\mathbf{s}$ (color blue online) undergoes a rotation, after which its spin becomes $\mathbf{s}'$ (color orange online). During this rotation, its spin components in the x, y and z direction change by Δs_x, Δs_y and Δs_z, respectively. This unitary evolution is implemented by acting on the system plus the particles of a reference frame (green box, online), composed of spin-$\frac{1}{2}$ particles that are divided into three groups: (i) the x-battery, composed of particles in the + eigenstates of s_y or s_z, (ii) the y-battery, composed of particles in the + eigenstates of s_x or s_z, and (iii) the z-battery, composed of particles in the + eigenstates of s_x or s_y. Up to arbitrary accuracy, the particles in the reference frame corresponding to the k-battery will absorb all and only the change Δs_k of the evolved spin system.

These results address a fundamental aspect of conservation laws in quantum theory. Furthermore, they are of particular importance in quantum thermodynamics, which has recently been extended to multiple non-commuting conserved quantities [10, 12–22], and where batteries for storing conserved quantities are of particular interest.

Separate batteries for spin-$\frac{1}{2}$ systems.—We consider the case of spin-$\frac{1}{2}$ particles, and the possibility of separating the different conserved components of angular momentum s_x, s_y and s_z (see Fig. 1). We will show that by considering a fixed reference frame composed of multiple spin-$\frac{1}{2}$ particles, and interacting with it in a particular way: (i) the total angular momentum of the system and reference frame is conserved, (ii) any unitary transformation can be implemented on the system with arbitrary precision, and (iii) any changes in the average angular momentum components of the system are stored in different parts of the reference system (up to arbitrarily small correction terms). Essentially, we can think of the reference system as being partitioned into three separate batteries, each of which stores a different component of angular momentum. We emphasize that the quantities being stored in the reference systems are the changes in expectation value of the non-commuting observables – the uncertainties in these observables within the reference system make the scenario compatible with the uncertainty principle.

The reference system must: (a) indicate the x, y and z directions in order to allow rotations of the system about these directions and (b) do this via a rotationally invariant interaction so that the total angular momentum is conserved. One way to construct a reference frame for the x-direction would be to prepare a number of spins all pointing in the x-direction (as in [9]). This could be used to implement a rotation of the system about x, but each spin in the reference frame would generally accumulate changes in both s_y and s_z, thus not separating the conserved quantities. The key intuition behind our approach is that there is an alternative way to define the x-direction. Instead of aligning each spin in the reference frame along the x-direction, we prepare pairs of spins pointing in the y and z direction, and implement a cross product via the interaction. In this way, each spin in the reference frame accumulates changes in only one component of spin, and thus the different conserved quantities can be separated.

Our result is formalised in the following theorem. Using lower case s for the spin of individual spin-$\frac{1}{2}$ particles and capital S for the spin of systems composed by many spins, we have the following. Let our system be a spin-$\frac{1}{2}$ particle, whose spin is denoted by $\mathbf{s}$, with components $\{s_k\}_{k=x,y,z}$. Let the total spin of the j-part of the reference frame (i.e. the part which is intended to store spin in the j-direction) be denoted by $\mathbf{S}^{(j)}$, $j = x, y, z$, and has components $\{S_k^{(j)}\}_{k:x,y,z}$; and finally, the total spin operator of the system and frame is denoted by $\mathbf{S}^{\text{tot}} = \mathbf{s} + \mathbf{S}^{(x)} + \mathbf{S}^{(y)} + \mathbf{S}^{(z)}$, and its components given by $S_k^{\text{tot}} = s_k + S_k^{(x)} + S_k^{(y)} + S_k^{(z)}$, $k = x, y, z$.

Theorem 1. *Let the system $\mathcal{S}$ be a spin-$\frac{1}{2}$ particle. Then for every $\epsilon > 0$ and $\delta > 0$, there exists a reference frame $\mathcal{R}$ (composed of a large num-*

ber of spin-half particles) with a fixed state $\rho_R = \rho_R^{(x)} \otimes \rho_R^{(y)} \otimes \rho_R^{(z)}$*, where* $\rho_R^{(k)}$ *is the state of the* k*-part of the reference system, such that for every unitary* U_S *on the system there exists a joint unitary* V *on the system and reference frame with the following properties:*

- Conservation: V *conserves all components of total angular momentum* $\boldsymbol{S}^{tot}$*, i.e.,* $[V, \boldsymbol{S}^{tot}] = 0$.

- Accuracy: V *effectively implements* U_S *on the system with precision* ϵ*, i.e., for any initial system state* ρ_S*,*

$$\left\| \mathrm{tr}_R \left\{ V\, \rho_S \otimes \rho_R\, V^\dagger \right\} - U_S \rho_S U_S^\dagger \right\|_1 \leq \epsilon. \quad (1)$$

- Separation: *Each component of angular momentum of the system is exchanged only with the corresponding part of the reference system, up to precision* δ*:*

$$\left| \Delta s_j + \Delta S_j^{(j)} \right| \leq \delta, \quad (2)$$

$$\left| \Delta S_j^{(k)} \right| \leq \delta \quad \text{if } j \neq k, \quad (3)$$

where Δs_j *is equal to the change in the average angular momentum of the system in the* j*-direction, and* $\Delta S_j^{(k)}$ *is equal to the change in the average angular momentum of the* k*-part of the reference system in the* j*-direction.*

Proof. The proof strategy is to first show how to implement a small rotation of the system around the x-axis, by using the system plus two spins in the reference frame, one polarised in the y-direction and one in the z-direction. We then extend this to small rotation about an arbitrary axis, by doing subsequent small rotations about the x, y and z-direction. Finally we repeat this many times to build up a general rotation the system.

The key is to consider the following operator acting on three spin-$\frac{1}{2}$ particles

$$T = \mathbf{s} \cdot (\mathbf{s}' \times \mathbf{s}'') = \sum_{j,k,\ell \in \{x,y,z\}} \epsilon_{jk\ell}\, s_j s'_k s''_\ell\,, \quad (4)$$

where $\mathbf{s}$, $\mathbf{s}'$ and $\mathbf{s}''$ are spin operators of the three particles. In the right-hand side of Eq. (4), the subindex j in the spin operators denotes the spin component in the j direction, and $\epsilon_{jk\ell}$ the Levi-Civita tensor. For simplicity we have set $\hbar = 1$, so each spin operator is equal to half the corresponding Pauli operator.

The operator T is invariant under rotations, as it is a "scalar", being defined as a dot product of two vectors, $\mathbf{s}$ and $(\mathbf{s}' \times \mathbf{s}'')$. We can now construct a unitary interaction between three qubits given by

$$V_\alpha = \exp\{-i \tfrac{4\alpha}{N} T\}. \quad (5)$$

Since T is invariant under rotations, it commutes with the total spin $\mathbf{S}^{\mathrm{tot}} = \mathbf{s} + \mathbf{s}' + \mathbf{s}''$. In particular this means that also V_α preserves the conserved quantities S_x, S_y and S_z, and hence satisfies $[V_\alpha, \mathbf{S}] = 0$ (see Supplementary Material [11, Sec. IIC]).

Let us define τ_j as the density matrix of a spin-$\frac{1}{2}$ particle pointing in the j direction (i.e. $\tau_j = \frac{I}{2} + s_j$); for example $\tau_z = |{\uparrow_z}\rangle\langle{\uparrow_z}| = \frac{I}{2} + s_z$,

Suppose that we want to implement a small rotation of the system about the x-direction, given by $U_{\alpha,x} = \exp\left(-i \frac{\alpha}{N} s_x\right)$. To do this we prepare the spins $\mathbf{s}'$ and $\mathbf{s}''$ in the density matrix $\tau'_y \otimes \tau''_z$ and act with V_α on the system and these two spins.

We show in the Supplemental Material [11, Sec. IA] that

$$\mathrm{tr}_R \left\{ V_\alpha\, \rho_S \otimes \tau'_y \otimes \tau''_z\, V_\alpha^\dagger \right\} = U_{\alpha,x}\, \rho_S\, U_{\alpha,x}^\dagger + \mathcal{O}\left(\tfrac{1}{N^2}\right). \quad (6)$$

In this way, the state $\tau'_y \otimes \tau''_z$ defines a reference frame for the direction x.

We now consider how the angular momentum in the reference system changes under this transformation.

$$\Delta s'_z = -\frac{\alpha}{N} \mathrm{tr}\left\{ s_y \rho_S \right\} + \mathcal{O}\left(\tfrac{1}{N^2}\right), \quad (7)$$

$$\Delta s''_y = \frac{\alpha}{N} \mathrm{tr}\left\{ s_z \rho_S \right\} + \mathcal{O}\left(\tfrac{1}{N^2}\right), \quad (8)$$

while all other components of the reference spins are left unchanged, up to $\mathcal{O}\left(\frac{1}{N^2}\right)$. These equations are proven in the Supplemental Material [11, Sec. IB]. Hence to leading order, the first reference system only picks up z-spin, and the second reference system only picks up y-spin. Therefore, in the context of the overall reference frame, $\mathbf{s}'$ belongs to the z-part, and $\mathbf{s}''$ to its y-part.

As total angular momentum is conserved, it follows that the change in the system's angular momentum obeys

$$\Delta s_x = \mathcal{O}\left(\tfrac{1}{N^2}\right), \quad \Delta s_y = -\Delta s''_y + \mathcal{O}\left(\tfrac{1}{N^2}\right), \quad \Delta s_z = -\Delta s'_z + \mathcal{O}\left(\tfrac{1}{N^2}\right). \quad (9)$$

Similarly, due to the cyclic symmetry of the spin operators, we can generate a small rotation of the system about the y-direction or z-direction by acting with V_α on the system and a reference frame in the state $\tau'_z \otimes \tau''_x$ or $\tau'_x \otimes \tau''_y$ respectively. In each case, the two components of angular momentum that change (perpendicular to the axis of rotation) are separated into the two different reference spins. For example, when performing a small y-rotation with the frame $\tau'_z \otimes \tau''_x$, to first order in $\frac{1}{N}$ only the x-spin of the first reference particle and the z-spin of the second reference particle are modified. Intuitively, neither reference system changes its spin-component parallel to the axis of rotation of the system (up to first order in $\frac{1}{N}$), and each reference spin changes its spin-component parallel to the direction it was originally pointing in only to second order in $\frac{1}{N}$, as it is originally at a maximum and the change in the perpendicular directions is of first order.

A small rotation about a general axis, given by the unitary $U_H = \exp\left(-i\frac{H}{N}\right)$ where $H = \sum_{k=x,y,z} \alpha_k \, s_k$ can be generated by performing three subsequent small rotations, around the x, y and z-directions as described above. In particular, we use 6 spin-$\frac{1}{2}$ particles, in the state

$$\tau_R = \tau_y^{(z)} \otimes \tau_z^{(y)} \otimes \tau_z^{(x)} \otimes \tau_x^{(z)} \otimes \tau_x^{(y)} \otimes \tau_y^{(x)}. \quad (10)$$

where the superscripts denote which part of the global reference frame the spins are in, i.e. which angular momentum component they will store. We then implement U_H by first applying V_{α_x} to the system and the first two reference spins, then V_{α_y} to the system and the next two reference spins, then V_{α_z} to the system and the last two reference spins. Following a similar approach to Eq. (6), we thereby obtain

$$\mathrm{tr}_R \left\{ V_{\alpha_z} V_{\alpha_y} V_{\alpha_x} \, \rho_S \otimes \tau_R \, V_{\alpha_x}^\dagger V_{\alpha_y}^\dagger V_{\alpha_z}^\dagger \right\} = U_H \, \rho_S \, U_H^\dagger + \mathcal{O}\left(\tfrac{1}{N^2}\right). \quad (11)$$

Finally, we iterate this procedure N times using a new set of six reference spins in the state τ_R each time. The overall reference frame is therefore $\rho_R = \tau_R^{\otimes N}$, which consists of $6N$ spins ($2N$ in each of the three parts).

In this way we will approximately implement the desired transformation $U_S = (U_H)^N = \exp(-iH)$, which is (up to a global phase) the most general unitary transformation on a spin-$\frac{1}{2}$ system.

Defining the full sequence of transformations by V, we note that as this is a sequence of V_α transformations, each of which conserve the three components of angular momentum, V will satisfy the *conservation* property $[V, \mathbf{S}^{\mathrm{tot}}] = 0$.

The error in implementing U_S is bounded by the sum of the errors from each step, giving a total error of $N\mathcal{O}\left(\frac{1}{N^2}\right) = \mathcal{O}\left(\frac{1}{N}\right)$ [9]. It follows that

$$\left\| \mathrm{tr}_R \left\{ V \, \rho_S \otimes \rho_R \, V^\dagger \right\} - U_S \rho_S U_S^\dagger \right\|_1 \leq \mathcal{O}\left(\tfrac{1}{N}\right), \quad (12)$$

which proves the *accuracy* property in Eq. (1) by suitably choosing a sufficiently large N according to the value of ϵ.

To prove the *separation* property, we use Eq. (9) and its equivalents for y and z rotations, together with the fact that there are only $2N$ spins in each part of the reference system, imply that

$$\left| \Delta s_j + \Delta s_j^{(j)} \right| \leq \mathcal{O}\left(\tfrac{1}{N}\right), \quad (13)$$

$$\left| \Delta s_j^{(k)} \right| \leq \mathcal{O}\left(\tfrac{1}{N}\right) \quad \text{if } j \neq k. \quad (14)$$

For any δ, we can therefore choose a sufficiently large N such that Eqs. (2) and (3) hold. Explicit bounds for the $\mathcal{O}\left(\frac{1}{N}\right)$ and $\mathcal{O}\left(\frac{1}{N^2}\right)$ terms in this section may be found in the Supplemental Material [11, Sec. IC]. □

These results can be extended to systems composed of any number of spin-$\frac{1}{2}$ particles, as we argue in the following. First, the above proof shows that we can implement any unitary on a single spin. Then, we can also implement interactions between two spins inside the system via an interacting unitary such as $\sqrt{SWAP}$ or $e^{-i\theta \, \mathbf{s}^1 \cdot \mathbf{s}^2}$. These commute with all extensive conserved quantities and do not require the use of a reference system. Thinking of our spin-$\frac{1}{2}$ systems as qubits, we know that the ability to perform all single-qubit unitaries plus any particular interacting two-qubit unitary is computationally universal [23]. Hence we can construct a circuit to approximately implement any unitary transformation on any number of spin-half systems, whilst storing any changes to angular momentum in different batteries.

Extracting the angular momentum components of an unknown spin.—An interesting possibility enabled by the above procedure is to take an unknown spin-$\frac{1}{2}$ state with average spin $\langle \mathbf{s} \rangle = (\langle s_x \rangle, \langle s_y \rangle, \langle s_z \rangle)$, and three other systems, and completely extract the different components of spin into the three systems (up to arbitrary accuracy). That is, we can perform a unitary transformation such

that, up to arbitrary accuracy, the spin-$\frac{1}{2}$ particle finally has average spin zero, and the average spin of the three systems has increased by $(\langle s_x\rangle, 0, 0)$, $(0, \langle s_y\rangle, 0)$, and $(0, 0, \langle s_z\rangle)$ respectively.

To do this, we use the x, y and z-parts of our reference frame as the three systems, and include two ancillary spin-$\frac{1}{2}$ particles, each in a maximally mixed state, in one of the systems. We then perform a unitary V on the entire state which is given by the circuit construction above, and which approximately implements a unitary

$$U = \sum_{n,m=0}^{1} X^n Z^m \otimes |n\rangle\langle n| \otimes |m\rangle\langle m| \qquad (15)$$

on the initial spin and the two maximally mixed spins, where X and Z are the Pauli unitaries on the system. The net effect of U is to completely decohere the state of the spin-$\frac{1}{2}$ particle and transform it into the maximally mixed state [24]. Because the ancillary spins are also left in maximally mixed states, all of the average angular momentum in the initial state of the spin must have been transferred to the three components of the reference frame, and hence to the three desired systems.

Conclusions.—We have shown that it is possible to perform an arbitrary unitary transformation on any number of spin-$\frac{1}{2}$ particles whilst respecting angular momentum conservation, in such a way that any changes in the three components of angular momentum are separated into different 'batteries'. Any errors in this procedure can be made arbitrarily small by making these batteries sufficiently large.

Our results also allow one to completely extract the different components of angular momentum of an unknown spin state into distinct systems (up to arbitrary precision). It would be interesting to investigate the ultimate limits of such a procedure, such as whether it can be made exact, and the smallest possible implementation.

The protocols we have described for spin-$\frac{1}{2}$ systems may be generalised to higher spins, when the objective is to implement a spatial rotation under total angular momentum conservation. However, such a higher-dimensional system has many potential conserved quantities and unitary transformations which are not related to spatial rotations. In the Supplemental Material [11, Sec. II] we show that, when the dimension of the Hilbert space is a power of 2, a complete set of conserved quantities can be constructed such that arbitrary unitary transformations on the system can be implemented while separating the changes in each conserved quantity in a different battery. An interesting open question is whether this can be achieved in arbitrary dimension, and which general sets of quantum observables allow for it.

Conservation laws are generally thought to arise from symmetries of nature. However, in some instances it could be interesting to treat various quantities as separately conserved. For example, one could consider the case of a Hamiltonian with terms corresponding to two different contributions to the energy which do not commute (e.g., potential and kinetic energy, or on-site and interaction terms in a spin chain), and constructing a reference frame which allows one to perform arbitrary transformations of the system whilst storing changes in these different contributions to the energy separately.

The study of quantum thermodynamics has recently been extended to other conserved quantities besides energy, in particular to non-commuting conserved quantities, and this result allows one to consider explicit batteries for the angular momentum. The fact that the different components of angular momentum can be separated in this way is particularly surprising given that they do not commute, and measurements of one component would disturb the others.

Finally, the use of the cross product technique is more than a simple technical development. It has, we believe, a deep conceptual meaning. It shows that in the realm of quantum mechanics one can build frames of reference in various ways, all equally good for acting as references (for specifying a direction in our case), but which have fundamentally different properties. This could have implications, for instance, in defining frames in the context of gravity – indeed, the initial development of special and general relativity started with the study of frames of reference. Here, actual frames of reference have back-reaction to the space-time surrounding them. Constructing quantum reference frames in different ways may allow us to alter this back reaction. Understanding this, as a matter of principle, may turn out to be important for progress in quantum gravity.

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* s.popescu@bristol.ac.uk
† ana.sainz@ug.edu.pl
‡ tony.short@bristol.ac.uk
§ andreas.winter@uab.cat

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